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Weighted Monotone Fock Space and A Brownian Motion with the Distribution of Bożejko-Leinert-Speicher

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Abstract. We construct on a weighted monotone Fock space Φ_w with weight sequence $w = (1, c, c^2, c^3, \cdots)$ an example of noncommutative Brownian motion $\{Q_t\}_{t\geq 0}$ such that the distribution $\mu_{s,t}$ of an increment $Q_t - Q_s$, $0 < s < t$, coincides with the distribution of Bożejko-Leinert-Speicher but that the process $\{Q_t\}_{t\geq 0}$ is not isomorphic to the c-free Brownian motion of Bożejko-Leinert-Speicher $\{\tilde{Q}_t\}_{t\geq 0}$.

1. Weighted Monotone Fock Space

A weighted monotone Fock space Φ_w is a deformation of the monotone Fock space Φ through a weight sequence $w = \{w_n\}_{n=0}^{\infty}$, $w_n > 0$. It is a special case of interacting Fock spaces [AcB, ALV]. The usual monotone Fock space corresponds to the case of trivial weight sequence $w_n := 1$, $n \geq 1$ [Lu, Mu1, Mu2]. Let us give the precise definitions.

Let $T = \mathbf{R}_+^*$ be the set of all strictly positive real numbers $s > 0$. Denote by Σ_n the set of all monotone sequences $\sigma = (s_n > s_{n-1} > \cdots > s_1)$ of length n from T, which are increasing to the left. For each $n \geq 1$, Σ_n is the measure space equipped with the (induced) Lebesgue measure $d\sigma$. $\Sigma_0 = \{\Lambda\}$ is the singleton consisting of the null sequence Λ with the point mass (= Dirac measure). Denote by $\mathcal{H}_n$ the complex L^2-space $\mathrm{L}^2(\Sigma_n)$ with a new inner product

$$< u|v > \;=\; w_n \int_{\Sigma_n} d\sigma\, \overline{u(\sigma)} v(\sigma) \qquad (u, v \in \mathcal{H}_n).$$

This Hilbert space $\mathcal{H}_n := (\mathrm{L}^2(\Sigma_n), w_n)$ is called the n-particle space. Then we put

$$\Phi_w \;:=\; \mathbf{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_n \oplus \cdots.$$

and call it a *weighted monotone Fock space* with a weight sequence $w = \{w_n\}_{n=0}^{\infty}$. Here we identified $\mathcal{H}_0$ with $\mathbf{C}$ through the identification of the function $\mathbf{1} : \Lambda \to 1$ with the unit 1 of $\mathbf{C}$. Also we assume that $w_0 = 1$. We denote by $\Omega := \mathbf{1}$ the vacuum vector ($\in \mathcal{H}_0$).

For each one-particle vector $f \in \mathcal{H}_1$, the creation operator δ_f^+ on Φ_w is defined, as the left multiplication operator, by

$$(\delta_f^+ u)(t{>}\sigma) \quad = \quad f(t)u(\sigma) \qquad (u = u(\sigma) \in \mathcal{H}_n).$$

The annihilation operator δ_f^- is defined as the adjoint of δ_f^+. For the vacuum vector Ω, we have $\delta_f^-\Omega = 0$. The concrete action of δ_f^- on $u \in \mathcal{H}_n$, $n \geq 1$, is given by

$$(\delta_f^- u)(\sigma) \quad = \quad \frac{w_{n+1}}{w_n} \int\limits_{t>\sigma} dt\, \overline{f(t)} u(t{>}\sigma).$$

So we put $r_n := \dfrac{w_n}{w_{n-1}}$, $n = 1, 2, 3, \cdots$, then a weight sequence $w = \{w_n\}_{n=0}^{\infty}$ corresponds to a sequence $\mathbf{r} = \{r_n\}_{n=1}^{\infty}$ in the bijective way:

$$w = (w_0, w_1, w_2, \cdots) \quad \longleftrightarrow \quad \mathbf{r} = (r_1, r_2, r_3, \cdots)$$

under the assumption $w_0 = 1$.

Let $\mathcal{A}_w$ be the C^*-algebra generated by the creation and annihilation operators $\{\delta_f^+, \delta_f^- | f \in \mathcal{H}_1\}$, and let $\phi_w(\cdot) =< \Omega | \cdot \Omega >$ be the vacuum state over $\mathcal{A}_w$. We will working on this C^*-probability space $(\mathcal{A}_w, \phi_w)$. We often use the short notation $< \cdot >:= \phi(\cdot)$ to mean the expectation w.r.t. a given state ϕ over a C^*-algebra.

2. Brwonian motion

For each $f \in \mathcal{H}_1$, the *canonical operator* Q_f is defined by $Q_f = \delta_f^+ + \delta_f^-$. By the specialization $f := \chi_{(0,t]}$ with $t \geq 0$, we obtain the creation process $D_t^+ = \delta_{\chi_{(0,t]}}^+$, the annihilation process $D_t^- = \delta_{\chi_{(0,t]}}^-$, and the pair of canonical processes $Q_t = D_t^+ + D_t^-$ and $P_t = \sqrt{-1}(D_t^+ - D_t^-)$. Here χ_I denotes the indicator function of an interval I on the real line. We are interested in the probability law of the canonical process $\{Q_t\}_{t\geq 0}$.

At first let us consider the independence structure of the process $\{Q_t\}_{t\geq 0}$. Let a C^*-probability space $(\mathcal{A}, \phi)$ and a stochastic process $\{X_t\}_{t\geq 0} \subset \mathcal{A}$ be given. Let $\mathcal{R}$ be the ring generated by all the semi-closed interval $(s,t]$ with $0 < s < t$. For each $I \in \mathcal{R}$, let $\mathcal{A}_I$ be the C^*-algebra generated by the increments $\{X_t - X_s | \ (s,t] \subset I\}$ supported in I. Then a process $\{X_t\}_{t\geq 0}$ is said to be a process with independent increments if, for each increasing finite sequence $I_1 < I_2 < \cdots < I_n$ of elements of $\mathcal{R}$, we have

$$\phi(A_1 A_2 \cdots A_n) \quad = \quad \phi(A_1)\phi(A_2)\cdots\phi(A_n)$$

for all $A_i \in \mathcal{A}_{I_i}$, $i = 1, 2, \cdots, n$. Of course $I < J$ means that $s < t$ for all $s \in I$ and all $t \in J$.

Proposition 2.1. *$\{Q_t\}_{t\geq 0}$ is a process with independent increments.*

This is a corollary of the following Propisition 2.2. For each $I \in \mathcal{R}$, let $\mathcal{A}_I^{(w)}$ be the C^*-algebra generated by $\{\delta_f^+, \delta_f^- | f \in \mathcal{H}_1; \mathrm{supp}(f) \subset I\}$. Then the following is easily shown.

Proposition 2.2. *Let $\{\mathcal{A}_I^{(w)} | I \in \mathcal{R}\}$ be the system of C^*-subalgebras of $(\mathcal{A}_w, \phi_w)$ defined above. Then for each increasing finite sequence $I_1 < I_2 < \cdots < I_n$ of elements of $\mathcal{R}$, we have*

$$\phi_w(A_1 A_2 \cdots A_n) = \phi_w(A_1)\phi_w(A_2)\cdots\phi_w(A_n)$$

for all $A_i \in \mathcal{A}_{I_i}^{(w)}$, $i = 1, 2, \cdots, n$.

Proposition 2.2 means that also the pair process $(Q_t, P_t)_{t\geq 0}$ is an independent increments process.

Proposition 2.3. *$\{Q_t\}_{t\geq 0}$ is a process with stationary increments, i.e.*

$$\phi_w((Q_{t+u} - Q_{s+u})^p) = \phi_w((Q_t - Q_s)^p)$$

for all $0 \leq s < t$, all $u > 0$ and all $p = 1, 2, 3, \cdots$.

The proof of Proposition 2.3 will become ovbious in the later. Now we know that $\{Q_t\}_{t\geq 0}$ is a process with independent and stationary increments. Besides $\{Q_t\}_{t\geq 0}$ is shown to be a scale-invariant process in the following sense.

Let $\{X_t\}_{t\geq 0} \subset \mathcal{A}$ (resp. $\{Y_t\}_{t\geq 0} \subset \mathcal{B}$) be a stochastic process on a C^*-probability space $(\mathcal{A}, \phi)$ (resp. $(\mathcal{B}, \psi)$). Put $\mathcal{A}_0 := C^*(\{X_t | t \in T\})$ and $\mathcal{B}_0 := C^*(\{Y_t | t \in T\})$ where $C^*(E)$ denotes the C^*-subalgebra generated by a subset E. Then a process $\{X_t\}_{t\geq 0}$ is said to be isomorphic to a process $\{Y_t\}_{t\geq 0}$ if there exists some C^*-isomorphism $\pi : \mathcal{A}_0 \to \mathcal{B}_0$ such that $\pi(X_t) = Y_t$ for all $t \geq 0$ and that $\psi(\pi(X)) = \phi(X)$ for all $X \in \mathcal{A}_0$. Under this definition, we have

Theorem 2.4. *For each $\lambda > 0$, the scaled process $\{Q_t' := \frac{1}{\sqrt{\lambda}} Q_{\lambda t}\}_{t\geq 0}$ is isomorphic to the original process $\{Q_t\}_{t\geq 0}$.*

Proof. For each fixed $\lambda > 0$, let us define a map $\Phi_w \xrightarrow{'} \Phi_w : u \xmapsto{'} u' = u_\lambda$ as follows. For each one-particle vector $f \in \mathcal{H}_1$, we put

$$f(\cdot) \xmapsto{'} f_\lambda(\cdot) = \frac{1}{\sqrt{\lambda}} f\left(\frac{1}{\lambda}\cdot\right).$$

Also for each n-particle vector $u \in \mathcal{H}_n$, we put

$$u(\cdot) \xmapsto{'} u_\lambda(\cdot) = \left(\frac{1}{\lambda}\right)^{\frac{n}{2}} u\left(\frac{1}{\lambda}\cdot\right).$$

Besides for the vacuum vector, we put

$$\Omega \overset{\prime}{\mapsto} \Omega.$$

Then this map $u \overset{\prime}{\mapsto} u' = u_\lambda$ defines a unitary operator on Φ_w, because we have

$$\begin{aligned} < u'|v' > &= w_n \int_\sigma d\sigma\, \overline{u'}(\sigma) v'(\sigma) \\ &= w_n \int_\sigma d\sigma\, \overline{\left(\frac{1}{\lambda}\right)^{\frac{n}{2}} u\left(\frac{1}{\lambda}\sigma\right)} \left(\frac{1}{\lambda}\right)^{\frac{n}{2}} v\left(\frac{1}{\lambda}\sigma\right) \\ &= w_n \int_\sigma \frac{1}{\lambda^n}\, d\sigma\, \overline{u\left(\frac{1}{\lambda}\sigma\right)}\, v\left(\frac{1}{\lambda}\sigma\right) \\ &= w_n \int_\tau d\tau\, \overline{u(\tau)}\, v(\tau) \\ &= < u|v >, \end{aligned}$$

where in the 4th equality we put $\tau = \frac{1}{\lambda}\sigma$, and used $d\sigma = \lambda^n d\tau$ because of $\sigma = \lambda\tau$. This unitary operator $\Phi_w \ni u \mapsto u' \in \Phi_w$ naturally induces the transformation of operators $T \mapsto T'$ as

$$\begin{array}{ccc} \Phi_w & \overset{\prime}{\longrightarrow} & \Phi_w \\ \downarrow T & & \downarrow T' \\ \Phi_w & \overset{\prime}{\longrightarrow} & \Phi_w \end{array} \qquad\qquad \begin{array}{ccc} u & \overset{\prime}{\longrightarrow} & u' \\ \downarrow T & & \downarrow T' \\ v & \overset{\prime}{\longrightarrow} & v' \end{array}$$

Then first we know $(\delta_f^+)' = \delta_{f_\lambda}^+$, because we have

$$\begin{aligned} ((\delta_f^+)'u')(t > \sigma) &= (\delta_f^+ u)'(t > \sigma) \\ &= \left(\frac{1}{\lambda}\right)^{\frac{n+1}{2}} (\delta_f^+ u)(\frac{1}{\lambda}(t > \sigma)) \\ &= \left(\frac{1}{\lambda}\right)^{\frac{1}{2}} f\left(\frac{1}{\lambda}t\right) \cdot \left(\frac{1}{\lambda}\right)^{\frac{n}{2}} u\left(\frac{1}{\lambda}\sigma\right) \\ &= f_\lambda(t) \cdot u_\lambda(\sigma) \\ &= (\delta_{f_\lambda}^+ u_\lambda)(t > \sigma) \\ &= (\delta_{f_\lambda}^+ u')(t > \sigma). \end{aligned}$$

Besides we know $(\delta_f^-)' = \delta_{f_\lambda}^-$, because we have

$$\begin{aligned} ((\delta_f^-)'u')(\sigma) &= (\delta_f^- u)'(\sigma) \\ &= \left(\frac{1}{\lambda}\right)^{\frac{n-1}{2}} (\delta_f^- u)(\frac{1}{\lambda}\sigma) \\ &= \left(\frac{1}{\lambda}\right)^{\frac{n-1}{2}} \frac{w_n}{w_{n-1}} \int_{t > \frac{1}{\lambda}\sigma} dt\, \overline{f}(t)\, u\left(t > \frac{1}{\lambda}\sigma\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{\lambda}\right)^{-1} \frac{w_n}{w_{n-1}} \int_{t>\frac{1}{\lambda}\sigma} \frac{1}{\lambda} d(\lambda t) \left(\frac{1}{\lambda}\right)^{\frac{1}{2}} \overline{f}(\frac{1}{\lambda}\cdot\lambda t)\cdot\left(\frac{1}{\lambda}\right)^{\frac{n}{2}} u\left(\frac{1}{\lambda}(\lambda t>\sigma)\right) \\
&= \frac{w_n}{w_{n-1}} \int_{s>\sigma} ds\, \overline{f_\lambda}(s)\cdot u_\lambda(s>\sigma) \\
&= (\delta^-_{\overline{f_\lambda}}\, u_\lambda)(\sigma) \\
&= (\delta^-_{\overline{f_\lambda}}\, u')(\sigma),
\end{aligned}
$$

where in the 5th equality we put $s=\lambda t$. So we have $(Q_f)' = Q_{f_\lambda}$. By the specialization $f := \chi_{[0,t)}$, we get

$$
(\chi_{[0,t)})_\lambda(s) \;=\; \frac{1}{\sqrt{\lambda}}\chi_{[0,t)}\left(\frac{1}{\lambda}\,s\right) \;=\; \frac{1}{\sqrt{\lambda}}\chi_{[0,\lambda t)}(s),
$$

and hence

$$
Q_{(\chi_{[0,t)})_\lambda} \;=\; Q_{\frac{1}{\sqrt{\lambda}}\chi_{[0,\lambda t)}} \;=\; \frac{1}{\sqrt{\lambda}}\, Q_{\chi_{[0,\lambda t)}}.
$$

So we obtain

$$
(Q_t)' \;=\; \frac{1}{\sqrt{\lambda}}\, Q_{\lambda t}.
$$

Besides it is easy to see that $T \mapsto T'$ is a C^*-algebra automorphism of $\mathcal{A}_w$ satisfying $\phi_w(T') = \phi_w(T)$. □

Corollary 2.5. *For each $t_1, t_2, \cdots, t_l \in T$, we have*

$$
< Q_{t_1} Q_{t_2} \cdots Q_{t_l} > = < Q'_{t_1} Q'_{t_2} \cdots Q'_{t_l} > .
$$

Proof. $T \overset{'}{\mapsto} T'$ is a C^*-algebra automorphism of $\mathcal{A}_w$ satisfying $\phi_w(T') = \phi_w(T)$. □

Proposition 2.6. $< Q_s Q_t > = \min\{s,t\}$ *for $s,t \in T$.*

By Propositions 2.1, 2.3 and 2.4, it is natural to interpret the process $\{Q_t\}_{t\geq 0}$ as a noncommutative analogue of Brownian motion.

3. Moments of Canonical Operators

Let a weighted monotone Fock space Φ_w with the weight sequence $w=\{w_n\}$ ($\leftrightarrow \mathbf{r} = \{r_n\}$) be given. In this section, we derive some recurrence relations concerning the moments of the distribution $\mu_{f,w} = \mu_{f,\mathbf{r}}$ of the canonical operator Q_f on Φ_w under the vacuum state ϕ_w. For simplicity we assume that $\| f \|_{L^2} = 1$. Put $\mu := \mu_{f,\mathbf{r}}$.

Let us treat the moments of μ:

$$
m_p \;=\; \phi_w(Q_f^p) \;=\; \int_{-\infty}^{+\infty} x^p d\mu, \qquad p = 0,1,2,3,\cdots.
$$

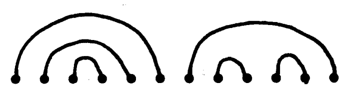
The pth moment m_p can be expanded as

$$\begin{aligned} m_p &= \phi(Q_f^p) = \phi((D_f^+ + D_f^-)^p) \\ &= \sum_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_p \in\{+,-\}} \phi(D_f^{\varepsilon_p}\cdots D_f^{\varepsilon_2}D_f^{\varepsilon_1}) \\ &= \sum_{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_p \in\{+,-\}} <\Omega|\, D_f^{\varepsilon_p}\cdots D_f^{\varepsilon_2}D_f^{\varepsilon_1}\Omega > . \end{aligned}$$

Besides it is easy to see that the contributing terms in the last expression are given by the sequences of signatures $(\varepsilon_p,\cdots,\varepsilon_2,\varepsilon_1)$ satisfying the following two conditions

$$\begin{cases} \#\{i\ |1\le i\le l,\ \varepsilon_i=+\} \ \ge\ \#\{i\ |1\le i\le l,\ \varepsilon_i=-\}, & l=1,\cdots,p, \\ \#\{i\ |1\le i\le p,\ \varepsilon_i=+\} \ =\ \#\{i\ |1\le i\le p,\ \varepsilon_i=-\}. \end{cases}$$

Such sequences $(\varepsilon_p,\cdots,\varepsilon_2,\varepsilon_1)$ correspond to the noncrossing pair partitions (=NCPP) of the p points set $\{p,p-1,\cdots,2,1\}$ in the bijective way. Besides the noncrossing pair partitions are identified with the noncrossing diagrams in the natural way. For example we have

$$(-,-,-,+,+,+,-,-,+,-,+,+) \longleftrightarrow$$


For a noncrossing diagram g which is corresponding to a sequence of signatures $(\varepsilon_p,\cdots,\varepsilon_2,\varepsilon_1)$, we put $V_{\mathbf r}(g) :=<\Omega|\, D_f^{\varepsilon_p}\cdots D_f^{\varepsilon_2}D_f^{\varepsilon_1}\Omega >$. Then we obtain a formula for the even moments m_{2k}

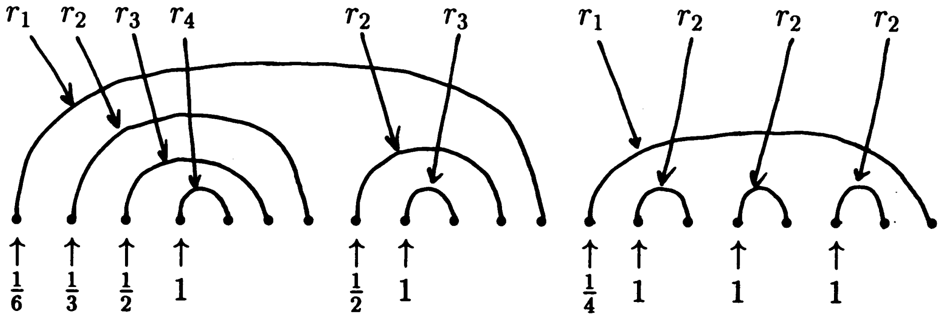
$$m_{2k} = \sum_{\substack{g:\ \text{NCPP of} \\ 2k\ \text{points set}}} V_{\mathbf r}(g)$$

Besides we can see that the following recurrence formula for $\langle\, g\,\rangle_{\mathbf r} := V_{\mathbf r}(g)$ hold.

Recurrence relations

$$\begin{aligned} &\text{(i)} & \langle\, \widehat{\ g_1\ }\ \widehat{\ g_2\ } \cdots \widehat{\ g_j\ }\,\rangle_{\mathbf r} &= \langle\, \widehat{\ g_1\ }\,\rangle_{\mathbf r}\,\langle\, \widehat{\ g_2\ }\,\rangle_{\mathbf r}\cdots\langle\, \widehat{\ g_j\ }\,\rangle_{\mathbf r} \\ &\text{(ii)} & \langle\, \widehat{\ g\ }\,\rangle_{\mathbf r} &= \frac{r_1}{|g|+1}\langle\, g\,\rangle_{\mathbf r'} \\ &\text{(iii)} & \langle\, \frown\,\rangle_{\mathbf r} &= r_1 \end{aligned}$$

Here $|g|$ denotes the number of lines in a diagram g. $\mathbf r'$ denotes the sequence obtained by the shift of $\mathbf r = (r_1,r_2,r_3,\cdots)$, that is, $\mathbf r' := (r_2,r_3,r_4,\cdots)$. For example, the following figure explains the rule for the calculation of $V_{\mathbf r}(g)$.



Let us write $m_p = m_p(\mathbf{r})$ to suggest explicitly the dependence on $\mathbf{r} = (r_1, r_2, r_3, \cdots)$. Using the recurrence relations for $V_{\mathbf{r}}(g)$, the $2k$th moment $m_{2k}(\mathbf{r})$ can be rewritten as

$$
\begin{aligned}
m_{2k}(\mathbf{r}) &= \sum_{\substack{g:\ \text{NCPP of} \\ 2k \text{ points set}}} V_{\mathbf{r}}(g) \\
&= \sum_{j=1}^{k} \sum_{\substack{g:\ \text{NCPP with} \\ \#\{\text{connected} \\ \text{components}\} = j}} V_{\mathbf{r}}(\widehat{g_1}\ \widehat{g_2} \cdots \widehat{g_j}) \\
&= \sum_{j=1}^{k} \sum_{\substack{k_1 + \cdots + k_j = k \\ k_1 \geq 1, \cdots, k_j \geq 1}} \sum_{\substack{|g_1| = k_1 - 1 \\ \cdots \\ |g_j| = k_j - 1}} V_{\mathbf{r}}(\widehat{g_1}) \cdots V_{\mathbf{r}}(\widehat{g_j}) \\
&= \sum_{j=1}^{k} \sum_{\substack{k_1 + \cdots + k_j = k \\ k_1 \geq 1, \cdots, k_j \geq 1}} \sum_{\substack{|g_1| = k_1 - 1 \\ \cdots \\ |g_j| = k_j - 1}} \frac{r_1}{k_1} V_{\mathbf{r}'}(g_1) \cdots \frac{r_1}{k_j} V_{\mathbf{r}'}(g_j) \\
&= \sum_{j=1}^{k} \sum_{\substack{k_1 + \cdots + k_j = k \\ k_1 \geq 1, \cdots, k_j \geq 1}} \left(\sum_{|g_1| = k_1 - 1} \frac{r_1}{k_1} V_{\mathbf{r}'}(g_1) \right) \cdots \left(\sum_{|g_j| = k_j - 1} \frac{r_1}{k_j} V_{\mathbf{r}'}(g_j) \right) \\
&= \sum_{j=1}^{k} \sum_{\substack{k_1 + \cdots + k_j = k \\ k_1 \geq 1, \cdots, k_j \geq 1}} \frac{r_1}{k_1} m_{2(k_1 - 1)}(\mathbf{r}') \cdot \cdots \cdot \frac{r_1}{k_j} m_{2(k_j - 1)}(\mathbf{r}').
\end{aligned}
$$

That is we get the recurrence formula

$$
m_{2k}(\mathbf{r}) = \sum_{j=1}^{k} \sum_{\substack{k_1 + \cdots + k_j = k \\ k_1 \geq 1, \cdots, k_j \geq 1}} \frac{r_1}{k_1} m_{2(k_1 - 1)}(\mathbf{r}') \cdot \cdots \cdot \frac{r_1}{k_j} m_{2(k_j - 1)}(\mathbf{r}'). \tag{3.1}
$$

Another form of recurrence formula is also useful:

$$
m_{2k}(\mathbf{r}) = \sum_{j=0}^{k-1} \frac{r_1}{j+1} m_{2j}(\mathbf{r}') \, m_{2(k-1-j)}(\mathbf{r}). \tag{3.2}
$$

Also we note here that $2k$th moment $m_{2k}(\mathbf{r})$ is a homogeneous polynomial of degree k in variables $r_1, r_2, \cdots, r_k$. So we have for each $c > 0$

$$m_{2k}(c\, r_1, c\, r_2, c\, r_3, \cdots, c\, r_n, \cdots) \;=\; c^k\, m_{2k}(r_1, r_2, r_3, \cdots, r_n, \cdots).$$

Let us derive a functional equation satisfied by the generating function $f(s) = f(s;\mathbf{r})$ for the even moments $\{m_{2k}(\mathbf{r})\}$ of the distribution $\mu = \mu_{\mathbf{r}} = \mu_{f,\mathbf{r}}$:

$$f(s;\mathbf{r}) \;=\; \sum_{k=0}^{\infty} m_{2k}(\mathbf{r}) s^k.$$

Using the recurrence relations for the moments (3.1), the generating function $f(s;\mathbf{r})$ can be rewritten as

$$\begin{aligned}
f(s;\mathbf{r}) &= \sum_{k=0}^{\infty} m_{2k}(\mathbf{r}) s^k \\
&= 1 + \sum_{k=1}^{\infty}\sum_{j=1}^{k} \sum_{\substack{k_1+\cdots+k_j=k \\ k_1\geq 1,\cdots,k_j\geq 1}} \frac{r_1}{k_1}\, m_{2(k_1-1)}(\mathbf{r}')\, s^{k_1} \cdots\cdots \frac{r_1}{k_j}\, m_{2(k_j-1)}(\mathbf{r}')\, s^{k_j} \\
&= 1 + \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} \frac{r_1}{k}\, m_{2(k-1)}(\mathbf{r}')\, s^k\right)^j .
\end{aligned}$$

Now we put $g(s;\mathbf{r}) := \sum_{k=1}^{\infty} \frac{r_1}{k}\, m_{2(k-1)}(\mathbf{r}')\, s^k$, then this quantity satisfies

$$f(s;\mathbf{r}) = \frac{1}{1-g(s;\mathbf{r})}.$$

Also this quantity $g(s;\mathbf{r})$ can be rewritten as

$$\begin{aligned}
g(s;\mathbf{r}) &= \sum_{k=1}^{\infty} \frac{r_1}{k}\, m_{2(k-1)}(\mathbf{r}')\, s^k \\
&= r_1 \sum_{k=1}^{\infty} \int_0^s ds\, m_{2(k-1)}(\mathbf{r}')\, s^{k-1} \\
&= r_1 \int_0^s ds \sum_{l=0}^{\infty} m_{2l}(\mathbf{r}')\, s^l \\
&= r_1 \int_0^s ds \sum_{k=0}^{\infty} m_{2k}(\mathbf{r}')\, s^k \\
&= r_1 \int_0^s ds\, f(s;\mathbf{r}').
\end{aligned}$$

So we get

$$g(s;\mathbf{r}) \;=\; r_1 \int_0^s ds\, f(s;\mathbf{r}'). \tag{3.3}$$

Therefore the moment generating function $f(s;\mathbf{r})$ satisfies the following functional equation:

$$\begin{cases} f(s;\mathbf{r}) &= \dfrac{1}{1 - r_1 \int_0^s ds\, f(s;\mathbf{r}')}, \\ f(0) &= 1. \end{cases}$$

4. An example – the distribution of Bożejko-Leinert-Speicher

In this section, we give an example of weighted monotone Fock space Φ_w such that the probability distribution μ_t of its associated Brownian motion $\{Q_t\}_{t\geq 0}$ can be explicitly obtained. This example corresponds to the weight sequence w given by

$$\mathbf{r} = (r_1, r_2, r_3, \cdots) := (1, c, c, c, \cdots).$$

In this case, the quantity $g(s;\mathbf{r})$ is given by

$$\begin{aligned} g(s;1,c,c,c,\cdots) &= 1 \cdot \int_0^s ds\, f(s;c,c,c,\cdots) \\ &= \int_0^s ds \sum_{k=0}^{\infty} m_{2k}(c,c,c,\cdots)s^k \end{aligned}$$

from (3.3). By the way, since in general the $2k$th moment $m_{2k}(r_1, r_2, r_3, \cdots)$ is a homogeneous polynomial of degree k in k variables $r_1, r_2, \cdots, r_k$, we have

$$m_{2k}(c,c,c,\cdots) = c^k\, m_{2k}(1,1,1,\cdots).$$

Note that $a_{2k} := m_{2k}(1,1,1,\cdots)$ is just the $2k$th moment of the arcsine law with mean 0 and variance 1 because the weight sequence $(1,1,1,\cdots)$ corresponds to the usual monotone Fock space [Mu1, Mu2]. Now $g(s;\mathbf{r})$ can be rewritten as

$$\begin{aligned} g(s;1,c,c,c,\cdots) &= \int_0^s ds \sum_{l=0}^{\infty} m_{2l}(1,1,1,\cdots)\,(cs)^l \\ &= \int_0^s ds\, f(cs;1,1,1,\cdots), \end{aligned}$$

where $f(s;1,1,1,\cdots)$ is just the generating function $a(s) := \frac{1}{\sqrt{1-2s}}$ for the even moments of the arcsine law. Hence we have

$$\begin{aligned} g(s;1,c,c,c,\cdots) &= \int_0^s ds\, a(cs) = \int_0^s ds \frac{1}{\sqrt{1-2cs}} \\ &= \left[-\frac{1}{c}(1-2cs)^{\frac{1}{2}}\right]_0^s = \frac{1}{c} - \frac{1}{c}(1-2cs)^{\frac{1}{2}}. \end{aligned}$$

Using the basic relation $f(s) = \frac{1}{1-g(s)}$, we obtain the explicit form of the generating function $f(s) = f(s; \mathbf{r})$ for the even moments of the distribution $\mu = \mu_{f,\mathbf{r}}$ associated to the Fock space Φ_w with the weight sequence $\mathbf{r} = (1, c, c, c, \cdots)$, as

$$f(s) \;=\; \frac{(c-1) - \sqrt{1-2cs}}{(c-2) + 2s}.$$

Then the Cauchy transform $G_\mu(z) = \int_{-\infty}^{+\infty} \frac{1}{z-\xi} d\mu(\xi)$ of the measure μ is given by

$$G_\mu(z) \;=\; \frac{1}{z} f\left(\frac{1}{z^2}\right) \;=\; \frac{(1-c)z + \sqrt{z^2 - 2c}}{(2-c)z^2 - 2}. \tag{4.1}$$

Here we remark that the expression (4.1) is obtained as the specialization $\alpha := 1$ & $\beta := \sqrt{\frac{c}{2}}$ of the Cauchy transform $G_{\nu_{\alpha,\beta}}(z)$ of the distribution of Bożejko-Leinert-Speicher $\nu_{\alpha,\beta}$, which is defined as follows [BLS]:

$$\begin{aligned}
\nu_{\alpha,\beta} &= \tilde{\nu}_{\alpha,\beta} + a\,(\delta_{x_1} + \delta_{x_2}), \\
d\tilde{\nu}_{\alpha,\beta}(x) &= \chi_{[-2\beta, 2\beta]}(x)\, \frac{1}{2\pi} \frac{\alpha^2 \sqrt{4\beta^2 - x^2}}{\alpha^4 - (\alpha^2 - \beta^2)x^2}\, dx, \qquad \text{(abs. conti. part)} \\
x_1 &= -\frac{\alpha^2}{\sqrt{\alpha^2 - \beta^2}}, \qquad x_2 = \frac{\alpha^2}{\sqrt{\alpha^2 - \beta^2}}, \qquad \text{(atomic part)} \\
a &= \begin{cases} \dfrac{1}{2}\dfrac{\alpha^2 - 2\beta^2}{\alpha^2 - \beta^2} & \left(0 \le \dfrac{\beta^2}{\alpha^2} \le \dfrac{1}{2}\right), \\ 0 & \left(\dfrac{1}{2} \le \dfrac{\beta^2}{\alpha^2}\right). \end{cases}
\end{aligned}$$

The Cauchy transform of $\nu_{\alpha,\beta}$ is known to be

$$G(z) \;=\; \frac{z(\frac{1}{2}\alpha^2 - \beta^2) + \frac{1}{2}\alpha^2\sqrt{z^2 - 4\beta^2}}{z^2(\alpha^2 - \beta^2) - \alpha^4}.$$

Hence we obtain the explicit form of the measure μ as follows.

$$\begin{aligned}
\mu &= \tilde{\mu} + b\,(\delta_{\xi_1} + \delta_{\xi_2}), \\
d\tilde{\mu}(x) &= \chi_{[-\sqrt{2c}, \sqrt{2c}]}(x)\, \frac{1}{\pi} \frac{\sqrt{2c - x^2}}{2 + (c-2)x^2}\, dx, \qquad \text{(abs. conti. part)} \\
\xi_1 &= -\sqrt{\frac{2}{2-c}}, \qquad \xi_2 = \sqrt{\frac{2}{2-c}}, \qquad \text{(atomic part)} \\
b &= \begin{cases} \dfrac{1-c}{2-c} & (0 \le c \le 1), \\ 0 & (1 \le c). \end{cases}
\end{aligned}$$

Now let us remove the assumption of $\| f \|_{L^2} = 1$. For general $f \in \mathcal{H}_1$, put $f = \| f \|_{L^2} \cdot u$ with $\| u \|_{L^2} = 1$, then we have $< Q_f^p > = (\| f \|_{L^2})^p < Q_u^p >$. Put $\mu_t := \mu_{\chi_{(0,t]}, \mathbf{r}}$, then we see that $\mu_t(dx) = \mu(\frac{dx}{\sqrt{t}})$, and hence μ_t equals to $\nu_{\sqrt{t}, \sqrt{\frac{ct}{2}}}$.

Since the distribution of Q_f dependes only on $\| f \|_{L^2}$, the distribution $\mu_{s,t}$ of an increment $Q_t - Q_s$ coinsides with μ_{t-s}.

After all we have

Proposition 4.1. *Let $\{Q_t\}_{t\geq 0}$ be the canonical process on a weighted monotone Fock space Φ_w with weight sequence $w = (1, c, c^2, c^3, \cdots)$. Then, under the vacuum state ϕ_w, the probability distribution $\mu_{s,t}$ of an increment $Q_t - Q_s$, $0 < s < t$, of the process $\{Q_t\}_{t\geq 0}$ is the distribution of Bożejko-Leinert-Speicher $\nu_{\alpha,\beta}$ with parameter $\alpha = \sqrt{t-s}$ and $\beta = \sqrt{\frac{c(t-s)}{2}}$.*

Remark 4.2. Note that, by the specializations $c = 1$ and $c = 2$ for μ, we get the arcsine law and the Wigner semicircle law, respectively.

$$\begin{cases} c = 1 \;\Rightarrow\; p(x) \;=\; \dfrac{1}{\pi}\dfrac{1}{\sqrt{2-x^2}} & \text{(arcsine law)} \\ \\ c = 2 \;\Rightarrow\; p(x) \;=\; \dfrac{1}{\pi}\sqrt{1-\left(\dfrac{x}{2}\right)^2} & \text{(Wigner semi-circle law)} \end{cases}$$

Remark 4.3. The distribution of Bożejko-Leinert-Speicher $\nu_{\alpha,\beta}$ was obtained in [BSp, BLS] as the central limit distribution in the c-free central limit theorem. Also the distribution of its associated Brownian motion $\{\tilde{Q}_t\}_{t\geq 0}$ is given by the distribution of Bożejko-Leinert-Speicher. We remark here that our Brownian motion $\{Q_t\}_{t\geq 0}$ is not isomorphic to the c-free Brownian motion of Bożejko-Speicher $\{\tilde{Q}_t\}_{t\geq 0}$ in [BSp] although they have the same distribution $\mu_t = \nu_{\alpha,\beta}$, for each time $t \geq 0$, with $\alpha = \sqrt{t}$ and $\beta = \sqrt{\frac{ct}{2}}$. The reason is that the correlation function of $\{Q_t\}_{t\geq 0}$ is different from the correlation function of $\{\tilde{Q}_t\}_{t\geq 0}$. For our Brownian motion $\{Q_t\}_{t\geq 0}$, the correaltion $< Q_sQ_tQ_tQ_s >$ is not symmetric in two variables s and t. Indeed, for $0 < s < t$, we have

$$< Q_sQ_tQ_tQ_s > = w_2 \left\{\frac{1}{2}s^2 + s(t-s)\right\} + w_1^2\, s^2, \tag{4.2}$$

whereas we have

$$< Q_tQ_sQ_sQ_t > = w_2 \left\{\frac{1}{2}s^2\right\} + w_1^2\, s^2. \tag{4.3}$$

Hence we know $< Q_sQ_tQ_tQ_s > \neq < Q_tQ_sQ_sQ_t >$, and recognize the non-symmetry in the roles played by the past s and the furture t ($0 \leq s < t$). On the other hand, for the c-free Brownian motion of Bożejko-Speicher $\{\tilde{Q}_t\}_{t\geq 0}$ in [BSp], it can be checked that

$$< \tilde{Q}_s\tilde{Q}_t\tilde{Q}_t\tilde{Q}_s > = < \tilde{Q}_t\tilde{Q}_s\tilde{Q}_s\tilde{Q}_t >$$

for $0 < s < t$. This concludes that $\{Q_t\}_{t\geq 0}$ is not isomorphic to $\{\tilde{Q}_t\}_{t\geq 0}$. Note that the expressions (4.2) and (4.3) hold for the Brownian motion $\{Q_t^{(w)}\}_{t\geq 0}$ of general weight sequence w. Now let $\Phi^{(\lambda)}$ be the interacting free Fock space over the one-particle space $\mathcal{H}_1 := L^2(\mathbf{R}_+)$, with the weight sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots)$, such that the probability measure $\mu^{(\lambda)}$ of its canonical operator $\bar{Q}_1$ coincides with μ_w. Such a sequence λ always exists (see [AcB]). Then we can check that

$$< \bar{Q}_s\bar{Q}_t\bar{Q}_t\bar{Q}_s > \;=\; < \bar{Q}_t\bar{Q}_s\bar{Q}_s\bar{Q}_t > .$$

for $0 < s < t$. So we observe that also, for each w, the Brownian motion $\{Q_t^{(w)}\}$ on Φ_w is not isomorphic to the Brownian motion $\{\bar{Q}_t^{(\lambda)}\}$ on the corresponding interacting free Fock space $\Phi^{(\lambda)}$ although they have the same distribution $\nu_{\sqrt{t},\sqrt{\frac{ct}{2}}}$ for each time $t \geq 0$.

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